Commensurations and Subgroups of Finite Index of Thompson’s Group $F$

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We determine the abstract commensurator $\text{Com}(F)$ of Thompson’s group $F$ and describe it in terms of piecewise linear homeomorphisms of the real line. We show $\text{Com}(F)$ is not finitely generated and determine which subgroups of finite index in $F$ are isomorphic to $F$. We also show that the natural map from the commensurator group to the quasi-isometry group of $F$ is injective.

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Introduction

Thompson’s groups have been extensively studied since their introduction by Thompson in the 1960s, despite the fact that Thompson’s account [7] appeared only in 1980. They have provided examples of infinite finitely presented simple groups, as well as some other interesting counterexamples in group theory (see for example, Brown and Geoghegan [3]). Cannon, Floyd and Parry [4] give an excellent introduction to Thompson’s groups where many of the basic results used below are proven carefully.

Automorphisms for Thompson’s group $F$ were studied by Brin [2], where a key theorem by McCleary and Rubin [6] is used to realize each automorphism as conjugation by a piecewise linear map. Here, we generalize from automorphisms to commensurations, which are isomorphisms between two subgroups of finite index. These form a group (under a natural equivalence relation involving passing to smaller yet still finite-index subgroups), called the commensurator group.

We classify finite-index subgroups of $F$, and then we extend Brin’s results from automorphisms to commensurations, again realizing every commensuration as conjugation by a piecewise linear homeomorphism of the real line. These maps exhibit a particular structure, satisfying an affinity condition in the neighborhood of $\infty$ which we use to find the algebraic structure of the commensurator of $F$. 
Commensurators have proven to be an effective tool for investigating quasi-isometries of a group to itself, and for effectively analyzing rigidity, particularly of lattices. In the case of $F$, the only quasi-isometries of $F$ known previously were automorphisms. This paper provides a wide array of examples of quasi-isometries, since all commensurations are quasi-isometries, and we prove in Section 5 that the commensurator group embeds into the quasi-isometry group in the case of $F$.

Our approach is algebraic, but we note that elements of the commensurator of $F$ can be represented by marked, infinite, eventually periodic, binary tree pair diagrams. We also note that recently Bleak and Wassink [1] have independently described the finite-index subgroups of $F$, using different methods.

The paper is organized as follows. In Section 1 we give the necessary definitions, and in Section 2 the first basic results for the finite-index subgroups of $F$. In Section 3 the main result about the commensurator is stated and proved, and in Section 4 its algebraic structure is given. The proof of the embedding of the commensurator group into the quasi-isometry group is given in Section 5.

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1 Definitions

Let $P$ denote the group of all homeomorphisms $f$ from $\mathbb{R}$ to itself that

1. are piecewise linear with a discrete (but possibly infinite) set of breakpoints (discontinuities of the derivative of $f$),
2. use only slopes that are integral powers of 2,
3. have their breakpoints in the set $\mathbb{Z}[\frac{1}{2}]$, and
4. satisfy $f(\mathbb{Z}[\frac{1}{2}]) \subset \mathbb{Z}[\frac{1}{2}]$. 

It is easy to check that each element \( f \) of \( P \) actually satisfies \( f(\mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}] \) and that \( P \) has a subgroup of index two which contains only the order preserving elements. We denote this subgroup by \( P_+ \). The quotient \( P/P_+ \) is generated by the image of the homeomorphism \( \tau : t \mapsto -t \).

Let \( f \in P \). We call \( f \) \textit{integrally affine} if \( f(t) = \xi t + p \) for some integer \( p \) and \( \xi \in \{ \pm 1 \} \).

We say \( f \) is \textit{periodically affine} if \( f(t + p) = f(t) + q \) for some non-zero \( p, q \in \mathbb{R} \) and \textit{integrally periodically affine} if \( p \) and \( q \) are integers. Note that all integrally affine maps are integrally periodically affine with \( q = \pm p \) depending on whether \( f \) is in \( P_+ \) or not.

When \( P \) is any of the above properties, then we call \( f \) \textit{eventually} \( P \) if \( f \) satisfies \( P \) for all \( t \in \mathbb{R} \) with \( |t| > M \) for some \( M > 0 \); here \( |t| \) denotes the absolute value of \( t \). For example, \( f \in P_+ \) is eventually integrally affine if there exist \( l, r \in \mathbb{Z}, M \in \mathbb{R}, M > 0 \), so that \( f(t) = t + r \) for all \( t > M \) and \( f(t) = t + l \) for all \( t < -M \). Notice that \( l \) and \( r \) may well be different.

It is well-known that Thompson’s group \( F \) is isomorphic to the subgroup of \( P_+ \) consisting of all eventually integrally affine elements (see [4]). It is easy to see that the commutator subgroup \( F' \) of \( F \) consists of all eventually trivial elements of \( P_+ \)(those where eventually \( f(t) = t \)). This group is denoted by \( BPL_2(\mathbb{R}) \) by Brin [2], where \( B \) stands for bounded support.

## 2 Finite-index Subgroups of \( F \)

Let \( f \) be an element of \( F \). Since \( f \) is eventually integrally affine, there are two integers \( l, r \) and a real number \( M > 0 \) such that \( f(t) = t + r \) for \( t > M \) and \( f(t) = t + l \) for \( t < -M \). The two numbers \( l \) and \( r \) are precisely the two components of the image of \( f \) in \( \mathbb{Z} \times \mathbb{Z} \) under the abelianization map. The subgroups of finite index of \( F \) are in one-to-one correspondence with those of its abelianization \( \mathbb{Z} \times \mathbb{Z} \) by the following result.

**Proposition 2.1** Let \( H \) be a subgroup of \( F \) of finite index. Then \( H \) contains \( F' \), the commutator subgroup of \( F \), and hence \( H \) is normal in \( F \). Moreover, \( H' = F' \).

**Proof** Since \( F \) is finitely generated, \( H \) has only finitely many conjugates in \( F \) and the intersection of all of them, \( K \) say, is normal and of finite index in \( F \). We consider \( K \cap F' \), which is thus normal and of finite index in \( F' \). Hence, since \( F' \) is simple and infinite, we conclude that \( K \cap F' = F' \) and \( F' \subset K \subset H \).
Hence $H$ is normal in $F$. The final claim follows from the fact that $H'$ is contained in $F'$ but also characteristic in $H$ and hence normal in $F$, whence $F' \subset H'$.  

From this fact we deduce that the finite-index subgroups of $F$ are in bijection with those of $\mathbb{Z} \times \mathbb{Z}$. There is a distinguished family among these—the subgroups $p\mathbb{Z} \times q\mathbb{Z}$. We denote by $[p, q], p, q \in \mathbb{Z}$, the preimage in $F$ under the abelianization homomorphism of the subgroup $p\mathbb{Z} \times q\mathbb{Z}$ of $\mathbb{Z} \times \mathbb{Z}$. Thus $F = [1, 1]$ and $F' = [0, 0]$.

### 3 The Commensurator Group

As mentioned before, a *commensuration* of a group $G$ is an isomorphism $\alpha: A \to B$, where $A$ and $B$ are subgroups of $G$ of finite index. Two commensurations $\alpha$ and $\beta$ are equivalent if they agree on some subgroup of finite index in $G$. In view of this, the product $\beta \circ \alpha$ of two commensurations $\alpha: A \to B$ and $\beta: C \to D$ is defined on $\alpha^{-1}(B \cap C)$. The set of all commensurations of $G$ modulo the above equivalence relation, together with this composition, forms a group called the *commensurator of $G$* which we denote by $\text{Com}(G)$. If $G$ is a subgroup of the group $H$, then the (relative) commensurator of $G$ in $H$, $\text{Com}_H(G)$, consists of all elements $h$ of $H$ for which $G \cap G^h$ has finite index in both $G$ and $G^h$; here $G^h = h^{-1}Gh$.

The main result of this paper is the following.

**Theorem 3.1** The commensurator of $F$ is isomorphic to $\text{Com}_P(F)$, which consists of all eventually integrally periodically affine elements (of $P$).

The strategy of the proof is to find a large group where $F$ is a subgroup, and in such a way that every commensuration can be seen as a conjugation by an element of the large group. The group $P$ plays this role in the case of $F$.

In order to explain this strategy, we need some definitions and one of the main results of McCleary and Rubin [6]. Let $(L, <)$ be a dense linear order. By *interval* we mean a nonempty open interval. A subgroup $G$ of $\text{Aut}(L)$ is *locally moving* if for every interval $I$ there exists a nontrivial element $g \in G$ which acts as the identity on $L \setminus I$. Finally, $G$ is *$n$-interval-transitive* if for every pair of sequences of intervals $I_1 < \cdots < I_n$ and $J_1 < \cdots < J_n$ there exists $g \in G$ such that $I_k \cap J_k \neq \emptyset$ for $1 \leq k \leq n$. Below, $\mathcal{L}$ denotes the Dedekind completion of $L$ which is assumed to have no endpoints.
Theorem 3.2 (McCleary–Rubin [6]) Assume $(L_i, <)$ is a dense linear order without endpoints and let $G_i \subset \text{Aut}(L_i)$ be locally moving and 2-interval transitive, $i = 1, 2$. Suppose that $\alpha : G_1 \to G_2$ is an isomorphism. Then there is a monotonic bijection $\tau : L_1 \to L_2$ which induces $\alpha$, that is, $g^\alpha = \tau^{-1}g\tau$ for every $g \in G_1$; and $\tau$ is unique.

Being locally moving and having 2-interval transitivity are local properties in the sense that a group inherits these from any of its subgroups.

Proof of Theorem 3.1 View $\mathbb{Z} [\frac{1}{2}]$ as a dense linear order and $F$ as the eventually integrally affine elements of $P_+$. Let $\alpha : A \to B$ be a commensuration of $F$. By Proposition 2.1, both $A$ and $B$ contain $F'$ which is (obviously) locally moving and 2-interval transitive (see [2, Lemma 2.1]). So Theorem 3.2 tells us that $\alpha$ is induced by conjugation with a unique element of $\text{Homeo} (\mathbb{R})$. This yields an injective homomorphism $\Psi : \text{Com}(F) \to \text{Homeo}(\mathbb{R})$.

Next, we show that the image of $\Psi$ is in fact contained in $P$. By Proposition 2.1, each commensuration of $F$ induces an automorphism of $F'$. In other words, the image of $\Psi$ is contained in $N_{\text{Homeo}(\mathbb{R})}(F')$, the normalizer of $F'$ in $\text{Homeo}(\mathbb{R})$. But this normalizer is equal to $P$ by Theorem 1 of Brin [2]. The existence and uniqueness statements in Theorem 3.2 now imply that $\Psi$ is an isomorphism between $\text{Com}(F)$ and $\text{Com}_P(F)$, which proves the first part of Theorem 3.1.

Let $\alpha \in \text{Com}(F)$ and choose positive integers $p$ and $q$ so large that $\alpha$ is defined on the subgroup $[p, q]$, that is $[p, q]^\alpha$, the image of $[p, q]$ under $\alpha$, is contained in $F$. By what was said above, we can view $\alpha$ as conjugation by an element of $P$. So for $f \in [p, q]$ we find $f^\alpha = \alpha^{-1}f\alpha$ to be eventually integrally affine. Suppose for a moment that $\alpha$ is order preserving and that $f(t) = t + kq$ for $t \gg 0$, where $k \in \mathbb{Z}$. Then

$$f^\alpha(t) = (\alpha \circ f \circ \alpha^{-1})(t) = \alpha(f(\alpha^{-1}(t))) = \alpha(\alpha^{-1}(t) + kq) = t + r$$

must hold for some $r \in \mathbb{Z}$. In other words, $\alpha^{-1}(t + r) = \alpha^{-1}(t) + s$ for some integers $r$ and $s$ and all $t \gg 0$. Since $f$ was arbitrary, we may assume that $k \neq 0$, which implies that $s \neq 0$, and hence also $r \neq 0$. Therefore $\alpha^{-1}$, and hence $\alpha$, must be integrally periodically affine near infinity. A similar calculation holds for $t \ll 0$ and also when $\alpha$ is order reversing. Consequently, each commensuration of $F$ must be eventually integrally periodically affine.

It remains to show that each eventually integrally periodically affine $\beta \in P$ induces a commensuration of $F$ by conjugation. Suppose $\beta(t + p) = \beta(t) + q$ for $t \gg 0$ and $\beta(t + p') = \beta(t) + q'$ for $t \ll 0$, with $p, q, p', q' \in \mathbb{Z} \setminus \{0\}$. Let $U = [p', p]$ if $\beta$ is
order preserving and set \( U = [p, p'] \) otherwise. Then for \( f \in U \), we have

\[
f^\beta(t) = \begin{cases} 
\beta(\beta^{-1}(t) + kp) = t + kq, & t \gg 0 \\
\beta(\beta^{-1}(t) + k'p') = t + k'q', & t \ll 0
\end{cases}
\]

where \( k, k' \in \mathbb{Z} \) depend on \( f \). Together with a similar argument for \( \beta^{-1} \) one easily sees that \( U^\beta = [q', q] \) or \([q, q']\), depending on whether \( \beta \) is order preserving or not. Theorem 3.1 is thus established.

We immediately obtain the following corollaries from this result.

**Corollary 3.3** A subgroup \( U \) of \( F \) of finite index is isomorphic to \( F \) if and only if \( U = [p, q] \) for some positive integers \( p \) and \( q \).

**Proof** Suppose \( U \) is a subgroup of finite index in \( F \). If \( U \) is isomorphic to \( F \), then there exists an eventually integrally periodically affine \( \alpha \in P \) with \( F^\alpha = U \) and calculations as above show that \( U \) must be of the form \([p, q]\). On the other hand, the final paragraph of the proof of the theorem read with \( p = p' = 1 \) shows that \([q', q]\) is isomorphic to \( F \) for every choice of positive integers \( q \) and \( q' \). This completes the proof.

Finally, since each subgroup of finite index in \( F \) contains \([p, q]\) for some positive integers \( p \) and \( q \) by Proposition 2.1, we have the following results.

**Corollary 3.4** Every finite-index subgroup of \( F \) is virtually \( F \).

**Corollary 3.5** A group is commensurable with \( F \) if and only if it is a finite extension of \( F \).

### 4 The Structure of Com(\( F \))

Descriptions of elements of \( \text{Com}(F) \) as conjugations in \( P \) allow us to study its structure as a group. An element \( \alpha \) of \( \text{Com}(F) \) is eventually integrally periodically affine, so there exist positive integers \( p, p', q, q' \) and a real number \( M \) such that

\[
\alpha(t + p) = \alpha(t) + q, \text{ for } t > M \\
\alpha(t + p') = \alpha(t) + q', \text{ for } t < -M.
\]

We need a lemma about affine functions, whose proof is elementary and left to the reader.
Lemma 4.1  Let \( f: \mathbb{R} \rightarrow \mathbb{R} \) be an integrally periodically affine map, and assume that there are integers \( i, i', j, j' \) such that for all \( t \in \mathbb{R} \) we have
\[
f(t + i) = f(t) + j \quad \text{and} \quad f(t + i') = f(t) + j'.
\]
Then we have
\[
f(t + r) = f(t) + s,
\]
where \( r = \gcd(i, i') \quad \text{and} \quad s = \gcd(j, j'). \)
Furthermore, we have
\[
\frac{i}{j} = \frac{i'}{j'}.
\]
From this lemma, we see that the integers \( p, p', q, q' \) for element of \( \text{Com}(F) \) depend only on the element.

We recall that \( \text{Com}(F) \) has a subgroup of index 2, denoted \( \text{Com}^+(F) \), formed by the commensurations induced by conjugations by piecewise-linear maps which preserve the orientation of \( \mathbb{R} \).

Proposition 4.2  There exists a surjective homomorphism \( \Phi: \text{Com}^+(F) \rightarrow \mathbb{Q}^* \times \mathbb{Q}^* \) defined by
\[
\Phi(f) = \left( \frac{p}{q}, \frac{p'}{q'} \right).
\]
Here \( \mathbb{Q}^* \) denotes the multiplicative group of the positive rational numbers.
The map is obviously well-defined due to the lemma above, and it is very easy to see that it is a homomorphism of groups. The two components of the map capture the behavior at both ends, eventually near \(-\infty\) and eventually near \(+\infty\). The two numbers \( p/q \) and \( p'/q' \) measure the “rate of growth” of the map at both ends.

A corollary of this result is that, as expected, \( \text{Com}(F) \) is infinitely generated.

5  Commensurations as Quasi-isometries

Let \( G \) be a finitely generated group. Quasi-isometries of \( G \) can be naturally composed, and there is a natural notion of equivalence class of quasi-isometries. Two quasi-isometries are considered equivalent if they are a bounded distance apart in the sense that \( f \) and \( g \) are considered equivalent if there exists a number \( M > 0 \) such that \( d(f(t), g(t)) \leq M \) for all \( t \) in \( G \).
Equivalence classes of quasi-isometries form elements of the group of quasi-isometries $QI(G)$ of $G$. It is well known that the commensurator group admits a map to the quasi-isometry group, since all commensurations give maps between finite index subgroups which are canonically quasi-isometric to the ambient group. The result we want to prove in this section is that for Thompson’s group $F$, this map is one-to-one.

**Theorem 5.1** The natural homomorphism $\text{Com}(F) \to QI(F)$ is injective.

We begin with an elementary lemma.

**Lemma 5.2** Given an element $\tau \in P$ which is different from the identity, there exist two intervals $I$ and $J$ of the real line, whose endpoints are dyadic integers, with $\tau(I) = J$, and such that $I \cap J = \emptyset$.

**Proof** The case when the slope of $\tau$ is always 1 or $-1$ is trivial. For a map $t \mapsto t + k$ has a small interval (of length less than $k$) whose image is disjoint from it. If $\tau = -\text{Id}$ the result is trivial.

If the slope is not constantly equal to 1, it has a piece with slope $\pm 2^i$ with $i \neq 0$. Assume without loss of generality (by possibly taking $\tau^{-1}$ instead of $\tau$) that $i > 0$. Hence there are two intervals $[a, b]$ and $[c, d]$ such that $\tau(a) = c$ and $\tau(b) = d$ and also $d - c = 2^i (b - a)$. It is possible that $[a, b]$ and $[c, d]$ overlap, but since $[c, d]$ is much larger than $[a, b]$ (at least twice the size), we can choose as $J$ a small interval inside $[c, d]$ which is disjoint from $[a, b]$. By construction, the preimage $I$ of $J$ is in $[a, b]$, and hence $I$ and $J$ are disjoint. \qed

**Proof of Theorem 5.1** We now take a nontrivial $\tau \in \text{Com}(F)$. By the previous lemma, there exist intervals $I$ and $J$ satisfying the conditions stated above and, in addition, that $I$, and hence $J$, have endpoints of the form $k/2^i$ and $(k + 1)/2^i$. We consider all elements of $F$ whose support (that is, the part where they are not the identity) is contained in $I$. Those elements form a subgroup which is isomorphic to $F$ itself. Let $f$ be one such element. Since its support is inside $I$, its image under the commensuration $\tau$, that is, $f^\tau = \tau \circ f \circ \tau^{-1}$, has support inside $J$.

Hence, the distance (inside $F$) from $f$ to $f^\tau$ is given by the distance from the identity to the element $f^\tau f^{-1}$. But this element has its support inside the disjoint union $I \cup J$, and the two parts are independent from each other (one given by $f$ and the other one by $f^\tau$). By work of Cleary and Taback [5], this subgroup—elements with support in $I \cup J$ which is a direct product of two clone subgroups in their terminology—is
quasi-isometrically embedded in $F$. Hence, we can take elements $f_n$ with support inside $I$ with arbitrarily large norm, and hence $f_n^*f_n^{-1}$ has also arbitrarily large norm. This proves that the image of $\tau$, a quasi-isometry, is not at bounded distance from the identity and the proof is complete.

References


